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Countable State Symbolic Dynamics

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1 Introduction

In this paper, we will investigate on countable state symbolic dynamics. In spite of success of getting many results for finite state symbolic dynamics, it is still almost open if the similar facts are also valid for countable state systems. Crucial difficulty to establish such analogous results may be non compactness of their sequence spaces. On the other hand, it is easy to find piecewise smooth dynamical systems whose symbolic dynamics provide countable state naturally. How strongly do their symbolic dynamics relate to themselves? How many properties do they reflect? Symbolic dynamics with finite state have been considered as a strong tool for studying smooth dynamical systems. The relation is explained as follows: The phase space of dynamcal system is partitioned into finite number of pieces, each piece labeled by a symbol; then by obserbing the itenerary of an orbit relative to the pieces of the partition, the orbit are coded into sequences of symbols. The motion of the dynamical system is then reflected by the shift map on the space of sequences. Even if dynamical systems are smooth only piecewisely, in some classes of such maps, (for example piecewise monotone maps on [0,1], piecewise linear Markov maps, etc) still their symbolic dynamice succeed to relate to themselves strongly. When does the maximal measure which is typical invariant measure in symbolic dynamics coincides with an invariant measure which is absolutely continuous with respect to Lebesgue measure?. In case of countable state, even to find nice class of piecewise smooth maps in the above sence is not easy. Futhermore countable state symbolic dynamics themselves are still not understandable. In [3], we discussed on some analogy of well-known facts in symbolic dynamics with finite state:

1 Sofic systems are one to one almost everywhere image of SFT's. Furthermore, it is well-known that,

2 A finite-to-one factor map is one-to-one almost everywhere if and only if it has a *right resolving block* (which is called sometime a *magic word*).

These facts seem to suggest a possibility of Markov approximation in a sence even in countable state symbolic dynamics. In this paper, we relate these analogies in case of countable state to our piecewise invertible systems. What kind of Markov approximation is possible? We will give partial answers to this question. In section 2, we will consider a relationship between piecewise invertible maps and their symbolic dynamics. In section 3, we will define a version of the Markov property which lead us to countable state Markov shifts and introduce a condition on piecewise invertible maps which leads us to countable state sofic shifts. We approximate such piecewise invertible systems by Markov systems along the line of Schweiger's technique ([4]). In section 4, We also have labelled graphs which give finite-to-one factor maps. This means that sofic shifts we will consider are finite to one images of SFT's. Furthermore we will find cylinders which give right resolving blocks (we call such cylinders *Markov cylinders*) by which we can construct piecewise invertible maps which may be considered as a kind of Markov approximation in a sence ([3]). In section 5, we will give a version of Takahashi's orbit basis by which we also obtain a kind of Markov apporoximation. The "finite range structure (abrr. FRS)" condition which is given in secton 3 will play important role to obtain our results of this paper.

2 Piecewise invertible systems and their symbolic dynamics

Let T be a transformation on a bounded domain $X(\subset \mathbf{R}^n)$ and $Q=\{X_a\}_{a\in I}$ be a countable partition of X satisfying the following conditions:

1. Q is the generating partition (i.e., $\vee_{m=0}^{\infty} T^{-m}Q$ is the partition into points).
2. Each X_a is a connected subset of X with piecewise smooth boundary, and X_a can not intersect any X_b with $b\neq a$.
3. T is a locally homeomorphism on each X_a.

We call such T a *piecewise invertible map* and call the triple (T, X, Q) a *piecewise invertible system*. If $int(X_{a_1}\cap T^{-1}X_{a_2}...\cap T^{-(n-1)}X_{a_n})\neq\emptyset$, then we say that the sequence $(a_1...a_n)$ is *admissible* and denote $X_{a_1}\cap T^{-1}X_{a_2}\cap...\cap T^{-(n-1)}X_{a_n}$ by $X_{a_1...a_n}$. We call $X_{a_1...a_n}$ *a cylinder of rank n*. $\mathscr{L}^n$ denotes the set of all cylinders of rand n. $|a_1...a_n|$ stands for the length of the sequence $(a_1...a_n)$. We denote $\mathscr{A}_n$ the set of all admissible sequences of length n and $\mathscr{A}=\cup_{n=1}^{\infty}\mathscr{A}_n$. Put $\mathscr{L}=\cup_{n=1}^{\infty}\mathscr{L}_n$. Let Σ^* be the set of all right semi-infinite sequences $(a_1a_2...a_n)$ satisfying $(a_1...a_n)\in\mathscr{A}$ for all $n>0$. For $(a_1a_2...a_n...)\in\mathscr{A}$, put $\rho(a_1a_2...)=\cap_{i=1}^{\infty}T^{-i}X_{a_{i+1}}$. It follows from the generator condition that $\rho(a_1a_2...)$ is at most single point of X. Put $\Sigma_\emptyset=\{(a_1a_2...)\in\Sigma^*: \rho(a_1a_2...)=\emptyset\}$. Define

$$\Sigma^{*\prime}=\bigcap_{i=0}^{\infty}\sigma^{*^{-i}}\{(a_1a_2...)\in\Sigma^*\setminus\Sigma_{\emptyset}:\rho(a_1a_2...)\in X_{a_1}\},$$

where σ^* is the left shift on Σ^*, i.e., $\sigma^*(a_1a_2...)=(a_2a_3...)$. We remark that Σ^* and $\Sigma^{*\prime}$ are σ^*-invariant.

Proposition 2.1 *$\rho:\Sigma^{*\prime}\to X$ is a bijective, continuous shift commuting map, i.e., $T\rho=\rho\sigma^{*\prime}$.*

We call the triple $(\Sigma^{*\prime}, \sigma^*, \rho)$ a realization of $(T, X, Q, \mathcal{U})$ or a symbolic dynamics of $(T, X, Q, \mathcal{U})$.

3 FRS and Markov approximation

We will define a version of the Markov property.

Definition We say that $X_{a_1...a_n}\in\mathscr{L}$ is *a Markov cylinder* if for $\forall X_{b(m)}\in\mathscr{L}^m$ satisfying $int(T^mX_{b(m)}\cap X_{a(n)})\neq\emptyset$, int $X_{a(n)}\subset T^mX_{b(m)}$.

Let $\mathscr{L}^n$ be the set of all Markov cylinders of rank n and put $\mathscr{L}=\cup_{i=1}^{\infty}\mathscr{L}^n$. Assume that

(C-1) T is onto (i.e., $X=\cup_{a\in I}TX_a$).

Then it follows from the condition 3 that T is a non-singular transformation with respect to Lebesgue measure λ of X. This fact and the local invertibility of T allow us to have the next result.

Proposition 3.1 *Let (T, X, Q) be a piecewise invertible system satisfying (C-1). For a Markov cylinder $X_{a_1...a_n}$, the following conditions are equivalent.*

(a) $(b_1...b_la_1...a_n)\in\mathscr{A}$.

(b) $T^{n+l}X_{b_1...b_la_1...a_n}=T^nX_{a_1...a_n}$ (λ mod 0).

(c) $T_lX_{b_1...b_l}\supseteq$int $X_{a_1...a_n}$.

Definition We say that T has the *k-Markov property* if $\mathscr{M}^k=\mathscr{L}^k$

Denote $\mathcal{U}^k=\{T^kX_{a_1...a_k}:(a_1...a_k)\in\mathscr{A}_k\}$ and put $\mathcal{U}=\cup_{k=1}^{\infty}\mathcal{U}^k$.

Remark A T satisfies the k-Markov property $\Leftrightarrow$ $\mathcal{U}=\mathcal{U}^k$. If T satifies the k-Markov property, then we call the quadraple $(T, X,$

Q, $\mathcal{U}$) a *piecewise invertible k-Markov system.* Its symbolic dynamics is exactly countable state 'k-Markov shift' which is defined in section 5.

Definition Let (T, X, Q) be a piecewise invertible system such that $\mathcal{U}$ consists of only finitely many subsets of X with positive Lebesgue measure. Then we say taht T guarantees a *finite range structure* (abrr. FRS) and we call the quadraple $(T, X, Q, \mathcal{U})$ a *piecewise invertible system with FRS.*

Remark B The subclass $\mathscr{M}$ of $\mathscr{L}$ has the "strong playback property", i. e., for $\forall X_{a(n)} \in \mathscr{M}$ and for $\forall X_{b(m)} \in \mathscr{L}$ such that $X_{b(m)a(n)} \in \mathscr{L}$, $X_{b(m)a(n)}$ also belongs to $\mathscr{M}$.

In fact, as the admissibility of the sequence $c(l) \cdot b(m)a(n)$ implies the admissibility of $c(l)b(m) \cdot a(n)$, the above assertioon is obtained immediately.

Remark C If $X_{a(n)}$ is a Markov cylinder, then for any $X_{b(m)} \in \mathscr{L}$ such that $X_{a(n)b(m)} \in \mathscr{L}$, $X_{a(n)b(m)}$ also belongs to $\mathscr{M}$.

Let us consider a new partition of X as follows: Put $\mathscr{B}_1 = \{X_b \in \mathscr{L}^1: X_b \in \mathscr{L}\}$ and $\mathscr{D}_1 = \{X_d \in \mathscr{L}^1: X_d \notin \mathscr{M}\}$. Denote $B_1 = \cup_{X_b \in \mathscr{B}_1} X_b$ and $D_1 = \cup_{X_d \in \mathscr{D}_1} X_d$. Next we devide D_1 into two disjoint subsets of D_1, B_2 and D_2 which consist of all cylinders of rank 2 belong to $\mathscr{M}^2$ and $\mathscr{L}^2 \backslash \mathscr{M}^2$ respectively. Put $\mathscr{B}_2 = \{X_{b_1 b_2} \in \mathscr{L}^2: X_{b_1} \in \mathscr{D}_1, X_{b_1 b_2} \in \mathscr{L}\}$ and $\mathscr{D}_2 = \{X_{d_1 da_2} \in \mathscr{L}^2: X_{d_1} \in \mathscr{D}_1, X_{d_1 d_2} \notin \mathscr{L}\}$. Inductively, we define

$$\mathscr{B}_i = \{X_{b(i)} \in \mathscr{L}^i: X_{d_1 \dots d_{i-1}} \in \mathscr{D}_{i-1}, X_{d(i)} \in \mathscr{L}\},$$
$$\mathscr{D}_i = \{X_{d(i)} \in \mathscr{L}^i: X_{d_1 \dots d_{i-1}} \in \mathscr{D}_{i-1}, X_{d(i)} \notin \mathscr{L}\}.$$

Put $B_i = \cup_{X_{b(i)} \in \mathscr{B}_1} X_{b(i)}$ and $D_i = \cup_{X_{d(i)} \in \mathscr{B}_1} X_{d(i)}$. For each $n > 0$, we have the following disjoint partition of X: $X = \cup_{i=1}^{n} B_i \cup D_n$. Let us consider a new countable state set I_M defined by $I_M=\cup_{j=1}^{\infty}\{(b_1\ldots b_j)\in\mathscr{A}: X_{b_1\ldots ba_j}\in\mathscr{B}_j\}$. Let us define a jump transformation T_M on $\cup_{j=1}^{\infty}\mathscr{B}_j$ by $T_M(x)=T^j(x)$ for $x\in\mathscr{B}_j$. Put $X_M=X\backslash(\cup_{m\geq 0}T_M^{-m}(\cap_{n\geq 0}D_n))$. Then T_M is a transformation of X_M.

Proposition 3.2 *If* $(T, X, Q, \mathcal{U})$ *is a piecewise invertible system with FRS, then* $(T_M, X_M, Q_M=\{X_a\}_{\alpha\in I_M}, \mathcal{U}_M)$ *is a piecewise invertible Markov system with FRS. (Here* $\mathcal{U}_M$ stands for the set of all range sets for (T_M, X_M, Q_M), *so* $\mathcal{U}_M\subset\mathcal{U}$.)

Lemma 3.1 $\cap_{n=1}^{\infty}D_n=\cup_{U\in\mathcal{U}}\partial U$.

As the number of elements of $\mathcal{U}$ is at most finite, we have:

Proposition 3.3 $X=X_M(\lambda\ mod\ 0)$

Theorem 3.1 T_M *has the (simple) Markov property, i.e., int* $(X_\alpha\cap T_M X_\gamma)\neq\emptyset$ *(where* $\alpha, \gamma\in I_M$*) implies* $T_M X_\gamma\supset X_\alpha$.

The following condition is needed to prove the main result in this section, and also plays an important role to construct orbit basis in the next discussion.

(C-2) If $X_{d_1\ldots da_{k-1}}\in\mathscr{D}_{k-1}$, $X_{d_k}\in\mathscr{D}_1$, and $X_{d_1\ldots d_k}\in\mathscr{S}^k$, then $X_{d_1\ldots d_k}\in\mathscr{D}_k$.

(Equivalently, $B_{n+1}=D_n\cap T^{-n}B_1$.)

Theorem 3.2 *Let* T *satisfy the conditions (C-1), (C-2). Then we can form the induced transformation* T_M of T with respect to $\mathscr{M}^1(=\mathscr{B}_1)$, *which satisfies* $T|_M\circ T^M=T_M\circ T|_M$.

We can apply the similar argument as the proof in [4] to our situation.

4 FRS and countable state sofic shifts

Our main purpose in this section is to show that FRS allows us to have a labelled graph with finitely many vertices and countably many edges, so that we can say that FRS leads to countable state sofic shift. Since we have a nice relation between the Markov approximation (T_M,

X_M, Q_M, $\mathcal{U}_M$) of (T, X, Q, $\mathcal{U}$) and its induced system T^M over $\mathscr{M}^1$, we can expect a relation between a tower over shift with respect to T_M and the symbolic dynamics of (T, X, Q, $\mathcal{U}$). We develop about this problem by uning Takahashi's orbit basis ([5]) in the next section.

Theorem 4.1 *Let* (T, X, $Q=\{X_a\}_{a\in I}$, $\mathcal{U}$) *be a piecewise invertible system with FRS. Suppose that* $\mathcal{U}=\{U_0, U_1,..., U_N\}$ *and* $X=U_0$. *Then there exists a countable state sofic shift which realizes* T *in the following sence: Define a directed graph whose vertex set is the finite set* $\mathcal{U}$, *arc set is the countable alphabet* I, *where there is an edge* c *from* U_i *to* U_j *if for* $\forall(a_1...a_l)$ *such that* $T^l X_{a_1...a_l}=U_i$, $X_{a_1...a_l c}\in\mathscr{S}$ *and* $T^{l+1} X_{a_1...a_l c} = U_j$.

1. *This labelled graph defines a one-sided edge SFT* σ *with countable alphabet and a one-block map* π: $\sigma\to\sigma'$ *where* σ' *is a subshift with the countable alphabet* I.
2. *Let* (σ', Σ') *be the one-sided sofic shift in the above, and for* $(a_0 a_1...)\in\Sigma'$ *put*

$$\rho(a_0a_1...)=\bigcap_{i=0}^{\infty} T^{-i}X_{a_i}.$$

Then the map ρ: $\Sigma'\to X$ *is defined a.e.* ρ *is a bijective continuous conjugacy map, i.e.,* $T\rho=\rho\sigma'$.

We use the following facts in the proof of Theorem 4.1.

Remark D $T(T^k X_{a_1...a_k}\cap X_c)=T^{k+1}X_{a_1...a_k c}$.

Remark E $(a_{-n}...a_{-1})(c_0...c_m)$ is admissible (i.e., $X_{a_{-n}...a_{-1}c_0...c_m}\in\mathscr{S}$) if and only if $\lambda(T^n X_{a_{-n}...a_{-1}}\cap X_{c_0...c_m})>0$. (It suffices to note the non-singularity of T with respect to λ.)

Lemma 4.1 $\{T^n X_{a_{-n}...a_{-1}}\}_{n>0}$ *is a monotone neasting sequence of subsets of* X *with positve Lebesgue measures, and* $\bigcap_{n>0}\{T^n X_{a_{-n}...a_{-1}}\}$ *is exactly some* $U_j\in\mathcal{U}$.

Proof

Note that

$T^{n+1}X_{a_{-(n+1)}a_{-n}\cdots a_{-1}} = T^{n+1}(X_{a_{-(n+1)}} \cap T^{-1}X_{a_{-n}\cdots a_{-1}}) = T^{n}(TX_{a_{-(n+1)}} \cap X_{a_{-n}\cdots a_{-1}})$.

It follows from the FRS condition that the following relation leads us to the desired result:

$$T^{n+1}X_{a_{-(n+1)}\cdots a_{-1}} \subseteqq T^{n+1}X_{a_{-(n+1)}} \cap T^{n}X_{a_{-n}\cdots a_{-1}} \subseteqq T^{n}X_{a_{-n}\cdots a_{-1}}. \square$$

Proof of Theorem 4.1

Let us consider the natural extension of T as two-sided shift on sequence space

$$\overline{\Sigma}^{*} = \{ (\ldots a_{-n} \ldots a_{-1};\ a_0 a_1 \ldots) : \exists\{x_i\}_{i\in Z}\ x_i \in X_{a_i} \text{ and } Tx_i = x_{i+1}\ (\forall i \in \mathbf{Z}) \}.$$

$\overline{\Sigma}^{*}_{(-)}$ denotes the set of all left infinite sequences appearing in points of $\overline{\Sigma}^{*}$. For $(\ldots a_{-n} \ldots a_{-1}) \in \overline{\Sigma}^{*}_{(-)}$, put

$$\mathscr{F}(\ldots a_{-n} \ldots a_{-1}) = \{ (c_0 c_1 \ldots) \in \Sigma^{*} : (\ldots a_{-n} \ldots a_{-1})(c_0 c_1 \ldots) \in \overline{\Sigma}^{*} \}.$$

Let us define an equivalence relation over $\overline{\Sigma}^{*}_{(-)}$ as follows:

$$(\ldots a_{-n} \ldots a_{-1}) \sim (\ldots b_{-n} \ldots b_{-1}) \iff \mathscr{F}(\ldots a_{-n} \ldots a_{-1}) = \mathscr{F}(\ldots b_{-n} \ldots b_{-1}).$$

From Remark E and Lemma 3.3, we can easily see that

$$\mathscr{F}(\ldots a_{-n} \ldots a_{-1}) = \mathscr{F}(\ldots b_{-n} \ldots b_{-1}) \iff \bigcap_{n>0} T^{n}X_{a_{-n}\cdots a_{-1}} = \bigcap_{n>0} T^{n}X_{b_{-n}\cdots b_{-1}}.$$

The FRS condition implies the finiteness of the number of equivalence classes so that (Σ^{*}, σ^{*}) satisfies Krieger's classical definition of sofic shift. So as well-known, the labelled graph defined in the theorem is exactly the graph G whose vertices are finitely many equivalence classes (we use the same notation $U_i \in \mathcal{U}$ for these classes), and whose edges are countably many alphabets of I. It follows from the assumption $U_0 = X$, that $\mathcal{U}$ coincide with the vertex set of G. Thus we can take (Σ^{*}, σ^{*}) for (Σ', σ'). We can see that a semi-infinite walk on the labelled graph gives just an itenerary of an orbit relative to the atoms of Q. More specifically, first we remark that one-sided sequences labelling semi-infinite paths in the labelled graph G belong to Σ^{*}. Assume further the existence of an edge d from U_j to U_k in G. This implies in original system $(T, X, Q, \mathcal{U})$ that,

$X_{a_1 \ldots a_l cd} \in \mathscr{L}$, and $T^{l+2} X_{a_1 \ldots a_l cd} = U_k$.

It follows from Remark D and Remark E that

$int(X_d \cap TX_c) \neq \emptyset$ if $int(X_d \cap U_j) \neq \emptyset$,

so that cd is admissible. Thus the admissibility rule of edges obtained from the labelled graph gives the admissibility rule of alphabets of I obtained from the motion of the original system. Conversely for $(a_0 a_1 \ldots) \in \Sigma^*$, if $\rho(a_0 a_1 \ldots) \in U_i$, then there is a semi-infinite path which is labelled by $(a_0 a_1 \ldots)$ and starts from U_i. Let us forget labels in G and let every edge be a different symbol. Then the new graph gives a one-sided edge SFT σ. The map π is obtained by reading off the labels of the edges (π: {edge}→{label}). Combining these consideration, we finish to prove 1 and 2.□

Remark F As mentioned in section 1, every Markov cylinder gives a right resolving block (magic word). More details is discussed in [3]. Markov cylinders also play important role in the next section.

5 Orbit bases for piecewise invertible systems with finite range structure

In this section, we discuss a relation between systems with FRS and shifts with orbit bases which was defined by Takahashi in [5]. First we will prepare some notation on shifts with countable states. Let (X, σ) be a one sided shift with a countabe state set I. For $(a_1 \ldots a_n) \in \mathscr{A}_n$, $[a_1 \ldots a_n]$ denotes the cylinder set

$$\{(x_1 x_2 \ldots) \in I^{\mathbb{N}} : (x_1 \ldots x_n) = (a_1 \ldots a_n)\}$$

For $V \subseteq \mathscr{A}$, denote $[V] = \cup_{(a_1 \ldots a_n) \in V} [a_1 \ldots a_n]$. We define for $V \subseteq \mathscr{A}$ *the Markov hull of V as follows*:

$$M(V)=\{(a_1a_2...)\in I^{\mathbf{N}}: (a_ia_{i+1}...)\in[V](\forall i\geq 1)\}.$$

The shift $(M(V), \sigma)$ for $V\subseteq\mathscr{A}_p$ is called *a p-Markov shift*. For a given one sided shift (X, σ), define $X^*=\{(a_n)_{n\in\mathbf{N}}=(...a_{-1}a_0a_1...)\in I^{\mathbf{Z}}: (a_{m+n})_{m\in\mathbf{N}}\in X(\forall n\in\mathbf{Z})\}$. Then (X^*, σ^*) is called *the natural extension* of (X, σ). Let $\pi: I^{\mathbf{Z}}\to I^{\mathbf{N}}$ be a natural projection i.e., $\pi(...a_{-1}a_0a_1...)=(a_0a_1...)$.

Let us define *an orbit basis* of a shift which is a version of Takahashi's definition in [5].

Definition

Let (X, σ) be a one sided shift over index set I. The shift (X, σ) is said to be admit *an orbit basis* if there exist a subset I_V *of* $\mathscr{A}$, *a positive integer* $r>0$, and a subset V of I_V^{r+1} satisfying the following conditions:

(1) For each $n\geq 1$, put $V_n=\{(\mathbf{b}_1...\mathbf{b}_n)\in I_V^n: (\mathbf{b}_1...\mathbf{b}_n)\in\mathscr{A}\}$. Then for $n>r$, $V_n=\{(\mathbf{b}_1...\mathbf{b}_n)\in I_V^n: (\mathbf{b}_i...\mathbf{b}_{i+r})\in V(1\leq i\leq n-r)\}$

(2) For each $x\in X$, there exists a sequence $(\mathbf{b}_n)_{n\geq 0}\in M(V)$ and some $0\leq i\leq|\mathbf{b}_0|$ such that
$$x=(\sigma^i\mathbf{b}_0)\mathbf{b}_1\mathbf{b}_2....$$

(3) Conversely the sequence defined by $(\sigma^i\mathbf{b}_0)\mathbf{b}_1\mathbf{b}_2...$ for some $(\mathbf{b}_n)_{n\geq 0}\in M(V)$ and $i>0$, belongs to X.

Let σ_V be a shift transformation on $I_V^{\mathbf{Z}}$, *so that* $(M(V), \sigma_V)$ is a r-Markov shift with structure set V. Let $(M(V)^*, \sigma_V^*)$ be a natural extension of $(M(V), \sigma_V)$ i.e.,

$$M(V)^*=\{(...\mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1...)\in I_V^{\mathbf{Z}}: (\mathbf{b}_n\mathbf{b}_{n+1}...)\in M(V)(\forall n\in\mathbf{Z})\},$$

which coincides with the set

$$\{(...\mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1...)\in I_V^{\mathbf{Z}}: (\mathbf{b}_n...\mathbf{b}_{n+r})\in V(\forall n\in\mathrm{Z})\}.$$

Let $\theta(\ldots\mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\ldots)=|\mathbf{b}_0|$ and put

$$Y=\{(\ldots\mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\ldots,\ n)\in M(V)^*\times\mathbf{N}:0\leq n\leq\theta(\ldots\mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\ldots)\}.$$

Define a map τ: $Y\rightarrow Y$ by

$$\tau(\ldots\mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\ldots,\ n)$$

$$=\begin{cases}(\ldots\mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\ldots,\ n+1) & \text{if}\,(\ldots\mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\ldots,\ n+1)\in Y\\ (\sigma_V^*(\ldots\mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\ldots),\ 0) & \text{otherwise.}\end{cases}$$

We call $(Y,\ \tau)$ *a tower over* $(M(V)^*,\ \sigma_V^*)$ *with the ceiling function* θ. Let us define a map i_0: $M(V)^*\rightarrow I^{\mathbf{Z}}$ *by*

$$i_0(\ldots\mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\ldots)(n)$$

$$=\begin{cases}\mathbf{b}_k(n-|\mathbf{b}_0|-\ldots-|\mathbf{b}_{k-1}|) & \text{if } 0\leq n\leq|\mathbf{b}_0|+\ldots|\mathbf{b}_k|\\ \mathbf{b}_k(n+|\mathbf{b}_{-1}|+\ldots+|\mathbf{b}_{-k}|) & \text{if } 0>n>-(|\mathbf{b}_{-1}|+\ldots+|\mathbf{b}_{-k}|).\end{cases}$$

Let i: $Y\rightarrow I^{\mathbf{Z}}$ be a map definded by

$$i(\ldots\mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\ldots,\ n)=\sigma^n i_0(\ldots\mathbf{b}_{-1}\mathbf{b}_0\mathbf{b}_1\ldots).$$

(4) The map i is a homeomorphism of Y into $(X^*,\ \sigma^*)$ such that $\pi i(Y)=X$ and $i\tau=\sigma^* i$.

We call $(I_V,\ V)$ an orbit basis of order r for $(X,\ \sigma)$. In particular, if $r=1$ we say that the orbit basis $(I_V,\ V)$ is *simple Markov*.

Remark G If $(X,\ \sigma)$ admits an orbit basis $(I_V,\ V)$, then there exists an onto map p such that $p\tau=\sigma p$. In fact we can take πi as p. We can say that in this sence $(X,\ \sigma)$ is a factor image of the tower over the $(r+1)$-Markov shift.

Next we will investigate on the existence of an orbit basis for our system with FRS.

Let $(T, X, Q=\{X_a\}_{a\in I}, \mathcal{U})$ be a piecewise invertible system with FRS.

Definition We call $X_{a_1\ldots a_n}\in\mathcal{M}^n$ *a special Markov cylinder* if $X_{a_i\ldots a_n}\in\mathcal{M}^c(\forall i\in\{2,\ldots n\})$.

Let $\mathcal{SM}^n$ be the set of all special Markov cylinders of length n.

Remark H From Remark B we have for $X_{a_1\ldots a_n}\in\mathcal{M}^c$, $X_{a_i\ldots a_n}\in\mathcal{M}^c(\forall i\in\{2,\ldots n\})$. From Remark C we also have $\mathcal{L}^n\backslash\mathcal{M}^n=\mathcal{D}_n$.

For each $n>0$, we have the following decomposition of $\mathcal{L}^n$ into three disjoint subsets:

$$\mathcal{L}^n=\mathcal{SM}^n\cup(\mathcal{M}^n\backslash\mathcal{SM}^n)\cup(\mathcal{M}^c\cap\mathcal{L}^n).$$

Denote $\mathcal{SM}=\cup_{n=1}^{\infty}\mathcal{SM}^n$, and let $I_{SM}=\{(a_1\ldots a_n)\in\mathcal{A}: X_{a_1\ldots a_n}\in\mathcal{SM}\}$.

Lemma 5.1 *For $X_{a_1\ldots a_n}\in\mathcal{M}^c$, there exists a cylinder $X_{b_1\ldots b_l}$ such that $X_{b_1\ldots b_l a_1\ldots a_n}\in\mathcal{M}$.*

Proof If we can not obtain such a sequence $(b_1\ldots b_l)\in\mathcal{A}$, then we have a strictly nesting sequence of range sets. In fact, if $X_{a_1\ldots a_n}\in\mathcal{M}$, there exists $(b_1\ldots b_l)\in\mathcal{A}$ such that $(b_1\ldots b_l a_1\ldots a_n)\in\mathcal{A}$ and $T^{n+l}X_{b_1\ldots b_l a_1\ldots a_n}\subset T^n X_{a_1\ldots a_n}$. If $X_{b_1\ldots b_l a_1}$ is also not a Markov cylinder, again we have some $(c_1\ldots c_j)\in\mathcal{A}$ such that $(c_1\ldots c_j b_1\ldots b_l a_1\ldots a_n)\in\mathcal{A}$ and

$$T^{n+l+j}X_{c(j)b(l)a(n)}\subset T^{n+l}X_{b(l)a(n)}\subset T_n X_{a(n)}.$$

Inductively, we can obtain an infinite nesting sequence of range sets: $T^n X_{a(n)}=U_{i_0}\supset U_{i_1}\supset\ldots$ This contradicts to our assumption "FRS".

From this lemma and the equality

$$\mathcal{M}^n\backslash\mathcal{SM}^n=\{X_{b_1\ldots b_n}\in\mathcal{M}^n: \exists b_i\in\mathcal{M}(2\leq i\leq n)\},$$

we have the following fact immediately.

Proposition 5.1 *For all $X_{a_1\ldots a_n}\in\mathcal{M}^c$, there exist $X_{d_1\ldots d_k}\in\mathcal{D}_k$* and $(\mathbf{b}_1\ldots$

$\mathbf{b}_m) \in I^m_{SM}$ *such that* $(a_1 \ldots a_n) = (d_1 \ldots d_k \mathbf{b}_1 \ldots \mathbf{b}_m) \in \mathscr{A}$

Even if $\mathcal{U}$ is a countable set, it is enough to add the following condition to obtain Proposition 5.1.

(C-3) $\cup_{U \in \mathcal{U}^*_M} U \supseteq D_1$, where $\mathcal{U}^*_M = \{T^n X_{a_1 \ldots a_n} : X_{a_1 \ldots a_n} \in \mathscr{M}\}$.

If there exists a cylinder $X_{a_1 \ldots a_n} \in \mathscr{M}$ with $T^n X_{a_1 \ldots a_n} = X$, the condition (C-3) is satisfied. In our example which is shown in [2], this is valid.

Proof For $X_{a_1 \ldots a_n} \in \mathscr{M}^c$, it follows from Remark H that $X_{a_1 \ldots a_n} \in \mathscr{D}_n$. The condition (C-3) implies that the existence of a $X_{b_1 \ldots b_k} \in \mathscr{M}$ such that *int* $(T^k X_{b_1 \ldots b_k} \cap X_{a_1 \ldots a_n}) \neq \emptyset$. Remark B allows us to have $X_{b_1 \ldots b_k a_1 \ldots a_n} \in \mathscr{M}$ and the desired decomposition as in the previous case.

Corollary 5.1 *For all* $X_{a_1 \ldots a_n} \in \mathscr{L}^n$, *we have an admissible sequence* $(\mathbf{b}_1 \ldots \mathbf{b}_m) \in I^m_{SM}$ *satysfying the following*: $(a_1 \ldots a_n) = ((\sigma^j \mathbf{b}_1) \mathbf{b}_2 \ldots \mathbf{b}_m)$, *where* $j \leq |\mathbf{b}_1|$.

Theorenm 5.1 *Under the condition (C-2),* $(T, X, Q, \mathcal{U})$ *with FRS admits the following orbit basis of order 1*:

$$I_V = I_{SM},$$
$$V = \{(\mathbf{b}_1 \mathbf{b}_2) \in I^2_{SM} : \mathbf{b}_1 \in Q(i) \text{ and } \mathbf{b}_2 \in \mathscr{M}(i) \text{ for } i \in \{0, 1, \ldots N\}\},$$

where $Q(i) = \{(a_1 \ldots a_n) \in \mathscr{A}^n : T^n X_{a_1 \cdots a_n} = U_i\}$ *and* $\mathscr{M}(i) = \{(a_1 \ldots a_n) \in \mathscr{A}^n : X_{a_1 \ldots a_n} \subseteq U_i\}$.

Lemma 5.2 *Let* $(a_1 \ldots a_n)$, $(b_1 \ldots b_m)$ *and* $(c_1 \ldots c_l)$ *belong to* $\mathscr{M}$. *If* $(a_1 \ldots a_n b_1 \ldots b_m)$ *and* $(b_1 \ldots b_m c_1 \ldots c_l)$ *are admissible, then* $(a_1 \ldots a_n b_1 \ldots b_m c_1 \ldots c_l) \in \mathscr{M}$.

Proof From Proposition 3.1 we have $T^{n+m} X_{a_1 \ldots a_n b_1 \ldots b_m} = T^m X_{b_1 \ldots b_m} \supseteq X_{c_1 \ldots c_l}$, so $(a_1 \ldots a_n b_1 \ldots b_m c_1 \ldots c_l) \in \mathscr{A}$ and also the strong playback property allows us to have $(a_1 \ldots a_n b_1 \ldots b_m c_1 \ldots c_l) \in \mathscr{M}$.

Remark I For $n \geq 2$, $I^n_{SM} \cap \mathscr{A} = \{(\mathbf{b}_1 \ldots \mathbf{b}_n) \in I^n_{SM} : \mathbf{b}_i \mathbf{b}_{i+1} \in V\}$.

Proof of Theorem For $(\mathbf{b}_n)_{n \geq 0}, (\mathbf{b}'_n)_{n \geq 0} \in \mathrm{M}(V)$, if there are some i,j with $0 \leq i \leq |\mathbf{b}_0|$, $0 \leq j \leq |\mathbf{b}'_0|$ satisfying

$$(\sigma^i \mathbf{b}_0)\,\mathbf{b}_1 \ldots = (\sigma^j \mathbf{b}'_0)\,\mathbf{b}'_1 \ldots,$$

then $\mathbf{b}_n = \mathbf{b}'_n (n \geq 1)$ and $|\mathbf{b}_0| - i = |\mathbf{b}'_0| - j$ and

$$(\mathbf{b}_{|\mathrm{b}_0|-i} \ldots \mathbf{b}_{|\mathrm{b}_0|}) = (\mathbf{b}'_{|\mathrm{b}'_0|-j} \ldots \mathbf{b}'_{|\mathrm{b}_0|}) \in \mathscr{D}_{|\mathrm{b}_0|-i}$$

This is an immediate consequence of the assumption (C-2). It follows from this fact that i is injective. All of other conditions in the definition of the orbit basis can be shown easily to use results we have already mentioned in this section.

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References

1 M. Yuri. Invariant measures for certain multi-dimensional maps. Preprint.

2 M. Yuri. Multi-dimensional maps with infinite invariant measures and countable state sofic shifts. Preprint.

3 M. Yuri. Subshifts of fnite range and countable state sofic shifts. Preprint.

4 F. Schweiger. Some remarks on ergodicity and invariant measures. *Michigan Math. J.* **22**(1975), 181-187.

5 Y. Takahashi. Shift with orbit basis and realization of one-dimensional maps. *Osaka J. Math.* **20**(1983), 599-629.